Workshop on Analysis, Control, and Operator Theory

IMB, University of Bordeaux

July, 6-8, 2021

Program of the conference

The schedules are given in Paris time (CEST).

Tuesday, 6/07/2021.

- 9:30-10:20: N. Nikolski, U. Bordeaux, France, *The best and the worst sign intermixings for frames and bases in* L^2.
- 10:30-11:10: Yu. Tomilov, IMPAN, Poland, *What can spectral theory do for (nonlinear) instability?*
- 11:10-11:35: Coffee break.
- 11:35-12:05: M.-A. Orsoni, U. Bordeaux, France, *Complex analysis and reachable space of parabolic equations.*
- 12:10-12:50: K. Pravda-Starov, U. Rennes 1, France, *Uncertainty principles and null-controllability of evolution equations enjoying Gelfand-Shilov smoothing effects.*
- 12:50-14:30: Lunch.
- 14:30-15:10: P. Thomas, U. Toulouse, France, *Invertibility threshold for Nevanlinna quotient algebras.*
- 15:15-15:45: J. Martin, U. Rennes 1, France, *Geometric conditions for the controllability of the free and harmonic Schrödinger equations.*
- 15:45-16:10: Coffee break.
- 16:10-16:40: S. Ervedoza, CNRS-U. Bordeaux, France, *Uniform null-controllability for a parabolic equation with discontinuous diffusion coefficients.*
- 16:45-17:25: A. Bufetov, CNRS, France; Steklov Institute, IITP RAS, Russia, *Determinantal point processes: quasi-symmetries and interpolation.*

Wednesday, 7/07/2021.

- 9:15-9:55: Yu. Belov, St. Petersburg State U., Russia, *Gabor analysis for rational functions.*
- 10:00-10:30: O. Ivrii, U. Tel-Aviv, Israel, *Entropy of universal covering maps.*
- 10:30-10:55: Coffee break.
- 10:55-11:35: A. Pushnitski, King's College, London, UK, *Inverse spectral theory for non-compact Hankel operators.*
- 11:40-12:20: A. Baranov, St. Petersburg State U., Russia, *Spaces of Cauchy transforms and localization of zeros.*

Thursday, 8/07/2021.

- 9:15-9:55: L. Rosier, U. du Littorale Côte d'Opale, France, *Exact controllability results of anisotropic 1D equations in spaces of analytic functions.*
- 10:00-10:30: L. Paunonen, Tampere U., Finland, *Non-uniform stability of damped contraction semigroups.*
- 10:30-10:45: Coffee break.
- 10:45-11:25: K. Beauchard, ENS Rennes, France, *On expansions for nonlinear systems, error estimates and convergence issues.*

- 11:30-12:10: L. Baratchart, INRIA, Sophia-Antipolis, France, *Stability of periodic delay systems and harmonic transfer function.*
- 12:10-14:05: Lunch.
- 14:05-14:45: K. Fedorovskiy, Moscow State U. - St. Petersburg State U., Russia, *Uniform approximation by polynomial solutions of elliptic equations and systems.*
- 14:50-15:20: E. Dubtsov, PDMI, St. Petersburg, Russia, *Calderón–Zygmund operators on RBMO.*
- 15:20-15:35: Coffee break.
- 15:35-16:15: K. Dyakonov, ICREA, U. Barcelona, Spain, *A Rudin–de Leeuw type theorem for functions with spectral gaps.*
- 16:20-17:00: N. Arcozzi, U. Bologna, Italy, *The Dirichlet space on the bi-disc.*
- 17:00-17:15: Coffee break.
- 17:15-17:55: P. Lissy, U. Paris-Dauphine, France, *A non-controllability result for the half-heat equation on the whole line based on the prolate spheroidal wave functions and its application to the Grushin equation.*
- 18:00-18:40: B. Wick, Washington University, St. Louis, US, *Singular integral operators on the Fock space.*

Titles and abstracts

Nicola Arcozzi, U. Bologna, Italy, *The Dirichlet space on the bi-disc.*

On a formal level, the Dirichlet space on the bidisc is the tensor product $\mathcal{D}(\mathbb{D}^2) = \mathcal{D}(\mathbb{D}) \otimes \mathcal{D}(\mathbb{D})$ of two copies of the classical holomorphic Dirichlet space. Multipliers and Carleson measures for the space were recently characterized, and the results have been extended to the three-disc, but not to higher powers. Underlying all this there is a new multi-parameter potential theory which is still in its infancy, and many basic problems await an answer. The talk reports on work by several authors: Pavel Mozolyako, Karl-Mikael Perfekt, Giulia Sarfatti, Irina Holmes, Alexander Volberg, Georgios Psaromiligkos, Pavel Zorin-Kranich, and the speaker.

Anton Baranov, St. Petersburg State U., Russia, *Spaces of Cauchy transforms and localization of zeros.*

We will discuss the spaces of functions which can be represented as the Cauchy integrals with L^2 data with respect to some fixed measure. These are Reproducing Kernel Hilbert spaces of functions analytic outside the support of the measure. They naturally appear in functional models for rank one perturbations of normal operators. In the talk we concentrate on the case of discrete measures in the complex plane with the following property: up to a finite number, all zeros of any Cauchy transform of the measure are localized near the support of the measure. We find several equivalent forms of this property and show that the parts of the support attracting zeros of Cauchy transforms are ordered by inclusion modulo finite sets.

The talk is based on joint works with E. Abakumov and Yu. Belov.

Laurent Baratchart, INRIA, Sophia-Antipolis, France, *Stability of periodic delay systems and harmonic transfer function.*

The Henry-Hale theorem says that a delay system with constant coefficients of the form $y(t) = \sum_{j=1}^{N} a_j y(t-\tau_j)$ is exponentially stable if and only if

$(I - \sum_{j=1}^{N} e^{-z\tau_j})^{-1}$ is analytic in $|z| > -\varepsilon$ for some $\varepsilon > 0$. We discuss an analog of this result when the a_j are periodic with Hölder-continuous derivative, saying that in this case exponential stability is equivalent to the analyticity of the so called harmonic transfer function for $|z| > -\varepsilon$, as a function valued in operators on $L^2(T)$ with T the unit circle.

This is joint work with S. Fueyo and J.B. Pomet.

Karine Beauchard, ENS Rennes, France, *On expansions for nonlinear systems, error estimates and convergence issues.*

Explicit formulas expressing the solution to non-autonomous differential equations are of great importance in many application domains such as control theory or numerical operator splitting. In particular, intrinsic formulas allowing to decouple time-dependent features from geometry-dependent features of the solution have been extensively studied.

First, we give a didactic review of classical expansions for formal linear differential equations, including the celebrated Magnus expansion (associated with coordinates of the first kind) and Sussmann's infinite product expansion (associated with coordinates of the second kind). Inspired by quantum mechanics, we introduce a new mixed expansion, designed to isolate the role of a time-invariant drift from the role of a time-varying perturbation.

Second, in the context of nonlinear ordinary differential equations driven by regular vector fields, we give rigorous proofs of error estimates between the exact solution and finite approximations of the formal expansions. In particular, we derive new estimates focusing on the role of time-varying perturbations. For scalar-input systems, we derive new estimates involving only a weak Sobolev norm of the input.

Third, we investigate the local convergence of these expansions. We recall known positive results for nilpotent dynamics and for linear dynamics. Nevertheless, we also exhibit arbitrarily small analytic vector fields for which the convergence of the Magnus expansion fails, even in very weak senses. We state an open problem concerning the convergence of Sussmann's infinite product expansion.

Eventually, we derive approximate direct intrinsic representations for the state and discuss their link with the choice of an appropriate change of coordinates.

This is a joint work with Jérémy Le Borgne, Frédéric Marbach.

Yurii Belov, St. Petersburg State U., Russia, *Gabor analysis for rational functions.*

Let g be a function in $L^2(\mathbb{R})$. By G_Λ, $\Lambda \subset R^2$ we denote the system of time-frequency shifts of g, $G_\Lambda = \{e^{2\pi i\omega x} g(x-t)\}_{(t,\omega)\in\Lambda}$.

A typical model set Λ is the rectangular lattice $\Lambda_{\alpha,\beta} := \alpha\mathbb{Z} \times \beta\mathbb{Z}$ and one of the basic problems of the Gabor analysis is the description of the frame set of g i.e., all pairs α, β such that $G_{\Lambda_{\alpha,\beta}}$ is a frame in $L^2(\mathbb{R})$.

It follows from the general theory that $\alpha\beta \leq 1$ is a necessary condition (we assume $\alpha, \beta > 0$, of course). Do all such α, β belong to the frame set of g?

Up to 2011 only few such functions g (up to translation, modulation, dilation and Fourier transform) were known. In 2011 K. Gröchenig and J. Stockler extended this class by including the totally positive functions of finite type (uncountable family yet depending on finite number of parameters) and later added the Gaussian finite type totally positive functions. We suggest another approach to the problem and prove that all Herglotz rational functions with imaginary poles also belong to this class. This approach also gives new results for general rational functions. In particular, we are able to confirm Daubechies conjecture for rational functions and irrational densities.

This is a joint work with Yu. Lyubarskii and A. Kulikov.

Alexander Bufetov, CNRS, France; Steklov Institute, IITP RAS, Russia, ***Determinantal point processes: quasi-symmetries and interpolation.***

The study of point processes, that is, random subsets of a Polish space, goes back at least to the 1662 work of John Graunt on mortality in London. Matrices whose entries are given by chance were studied by Ronald Fisher in 1915 and John Wishart in 1928 and used by Freeman Dyson who in 1962 observed that "the statistical theory (...) will describe the degree of irregularity (...) expected to occur in any nucleus".

The Weyl character formula implies that the correlation functions for the eigenvalues of a Haar-random unitary matrix have determinantal form – the Ginibre-Mehta theorem, – and in 1973 Odile Macchì started the systematic study of point processes whose correlation functions are given by determinants.

This level of abstraction has proved very fruitful: on the one hand, examples of determinantal point processes arise in diverse areas such as asymptotic combinatorics (Burton-Pemantle, Benjamini-Lyons-Peres-Schramm, Baik-Deift-Johansson, Borodin-Okounkov-Olshanski), representation theory of infinite-dimensional groups (Olshanski, Borodin-Olshanski), random series (Hough-Krishnapur-Peres-Virág) and, of course, random matrices; on the other hand, the general theory of determinantal point processes includes limit theorems (Soshnikov), a characterization of Palm measures (Shirai-Takahashi), the Kolmogorov as well as the Bernoulli property (Lyons, Lyons-Steif), and rigidity (Ghosh, Ghosh-Peres).

In this talk, the correlation kernels of our determinantal point processes will be assumed to induce orthogonal projections: for example, the sine-kernel of Dyson induces the projection onto the Paley-Wiener space of functions whose Fourier transform is supported on the unit interval, while the Bessel kernel of Tracy and Widom induces the orthogonal projection onto the subspace of square-integrable functions whose Hankel transform is supported on the unit interval.

What is the relation between the point process and the Hilbert space that governs it? Extending earlier work of Lyons and Ghosh, in joint work with Qiu and Shamov it is proved that almost every realization of a determinantal point process is a uniqueness set for the underlying Hilbert space. For the sine-process, almost every realization with one particle removed is a complete and minimal set for the Paley-Wiener space, whereas if two particles are removed, then one obtains a zero set for the Paley-Wiener space. Quasi-invariance of the sine-process under compactly supported diffeomorphisms of the line plays a key rôle.

The 1933 Kotelnikov theorem samples a Paley-Wiener function from its restriction onto the integers. How to reconstruct a Paley-Wiener function from a realization of the sine-process? In joint work with Borichev and Klimenko, it is proved that if a Paley-Wiener function decays at infinity as a sufficiently high negative power of the distance to the origin, then the Lagrange interpolation formula yields the desired reconstruction. Similar results are also obtained for the Airy kernel, the Bessel kernel and the Ginibre kernel of orthogonal projection onto the Fock space.

In joint work with Qiu, the Patterson-Sullivan construction is used to interpolate Bergman functions from a realization of the determinantal point process with the Bessel kernel, in other words, by the Peres-Virág theorem, the zero set of a random series with independent complex Gaussian entries. The invariance of the zero set under the isometries of the Lobachevsky plane plays a key rôle.

Evgenii Dubtsov, PDMI, St. Petersburg, Russia, ***Calderón–Zygmund operators on RBMO.***

Let μ be an n-dimensional finite positive measure on $\mathbb{R}^m$. We obtain a $T1$ condition sufficient for the boundedness of Calderón–Zygmund operators on $\mathrm{RBMO}(\mu)$, the regular BMO space of Tolsa.

This is a joint work with Andrei V. Vasin.

Konstantin Dyakonov, ICREA, U. Barcelona, Spain, ***A Rudin–de Leeuw type theorem for functions with spectral gaps.***

Our starting point is a theorem of de Leeuw and Rudin that describes the extreme points of the unit ball in the Hardy space H^1. We extend this result to subspaces of H^1 formed by functions with smaller spectra. More precisely, given a finite set E of positive integers, we prove a Rudin–de Leeuw type theorem for the unit ball of H^1_E, the space of functions $f \in H^1$ whose Fourier coefficients $\widehat{f}(k)$ vanish for all $k \in E$.

Sylvain Ervedoza, CNRS-U. Bordeaux, France, ***Uniform null-controllability for a parabolic equation with discontinuous diffusion coefficients.***

In this talk, I will present a study of the null-controllability of a heat equation in a domain composed of two media of different constant conductivities. In particular, I will be interested in the behavior of the system when the conductivity of the medium on which the control does not act goes to infinity, corresponding at the limit to a perfectly conductive medium. In that case, and under suitable geometric conditions, we obtain a uniform null-controllability result. Our strategy is based on the analysis of the controllability of the corresponding wave operators and the transmutation technique, which explains the geometric conditions.

This is a joint work with Jérémi Dardé (Toulouse) and Roberto Morales (Valparaiso).

Konstantin Fedorovskiy, Moscow State U. - St. Petersburg State U., Russia, ***Uniform approximation by polynomial solutions of elliptic equations and systems.***

We will deal with uniform approximations of functions by polynomial solutions of second order homogeneous elliptic equations with constant complex coefficients and by solutions of systems of such equations on compact sets in the complex plane. In this topic there are several interesting questions having an independent interest. The first one is the question on approximation by solutions of strongly elliptic equations of the type in question. In this topic there is an important open conjecture that the classical Walsh–Lebesgue criterion for uniform approximation by harmonic polynomials remains valid for the case of uniform approximation by polynomial solutions of every strongly elliptic equation of the type mentioned above. The second question is related with the not strongly elliptic case. In this case the approximation criteria are obtained for a wide class of compact sets, and the criteria obtained are stated in terms of Nevanlinna and $\mathcal{L}$-special domains. The studies of these special analytic characteristics of compact sets is the topical problem in the subject in question. The concept of a Nevanlinna domain (it corresponds to the case of approximation by bianalytic polynomials) is the most known and most deeply studied characteristic of sets of such kind, while the concept of a $\mathcal{L}$-special domain is quite poorly studied. All problems mentioned above need to be considered in the general context of approximation by polynomial solutions of systems of second order elliptic equations with constant coefficients (since elliptic equations with complex coefficients correspond to skew-symmetric systems of the type in question). In the talk we will present several recent results concerning all these approximation problems and discuss some open questions and conjectures related to them.

Oleg Ivrii, U. Tel-Aviv, Israel, ***Entropy of universal covering maps.***

Let P be a finite set of points in the unit disk not containing the origin and $\mathcal{U}_P : \mathbb{D} \to \mathbb{D} \setminus P$ be the universal covering map normalized so that $\mathcal{U}_P(0) = 0$

and $\mathcal{U}_P'(0) = 0$. It is well known that the Lebesgue measure on the unit circle is invariant under $\mathcal{U}_P$. In this talk, I will show that its measure-theoretic entropy is equal to

$$\frac{1}{2\pi}\int_{|z|=1} \log|\mathcal{U}_P'(z)|\,|dz| = \sum_{p\in P}\log\frac{1}{|p|} - \sum_{z\in Z}\log\frac{1}{|z|}$$

where Z is the set of zeros of $\mathcal{U}_P$ other than the origin. I credit the above formula to Pommerenke who proved an equivalent statement in his paper “On Greens functions of Fuchsian groups.”

Pierre Lissy, U. Paris-Dauphine, France, ***A non-controllability result for the half-heat equation on the whole line based on the prolate spheroidal wave functions and its application to the Grushin equation.***

In this talk I give two non-controllability results. The first one concerns the half-heat equation on the real line, when the control is supported outside of a segment of arbitrary small size. The second one concerns the Grushin equation on the plane, where the control is located outside of a horizontal band of arbitrary small thickness. After an appropriate duality argument, the question reduces to disproving some appropriate observability inequalities for the free half-heat and Grushin equations, which intuitively means that we want to construct solutions that remains “very concentrated” in space outside of the control region, at any positive time. I will explain how one can use some special functions, the Prolate Spheroidal Wave Functions (PSWF), that saturate a very special case of the Logvinenko-Sereda uncertainty principle, as initial conditions (up to a translation in the Fourier space) in order to disprove the observability inequality for the heat equation. Then, I will explain how my construction naturally leads to the result for the Grushin equation.

Jérémie Martin, U. Rennes 1, France, ***Geometric conditions for the controllability of the free and harmonic Schrödinger equations.***

In this talk, we will discuss geometric conditions ensuring the exact controllability of the free and harmonic Schrödinger equations. Necessary and sufficient conditions in one dimension will be presented and will be deduced from an uncertainty principle established by Kovrijkine and an infinite dimensional version of the Hautus test for Hermite functions. We will also present necessary conditions and sufficient conditions in higher dimensions for these equations.

This is a joint work with Karel Pravda-Starov (Université de Rennes 1).

Nikolai Nikolski, U. Bordeaux, France, ***The best and the worst sign intermixings for frames and bases in L^2.***

The sign intermixing of a real function $u(x)$ with the zero mean $\int u = 0$ is measured with the minimal transportation cost of positive masses $u^+(x)$ to the negative ones $u^-(x)$ (the Kantorovich-Rubinstein-Wasserstein norm $\|u\|_{KR}$). The smaller is the KR-norm, the better is the intermixing. Sharp summability exponents $\sum \|u_n\|_{Kr}^{\alpha} < \infty$ are found for the "best" and "worst" orthonormal bases, frames and/or Bessel sequences in $L^2((0,1)^d)$, $d = 1, 2, \ldots$.

The talk is a report on a joint project with A. Volberg.

Marcu-Antone Orsoni, U. Bordeaux, France, ***Complex analysis and reachable space of parabolic equations.***

The description of the reachable space of systems governed by PDE's is a central question in control theory. In this talk, I will discuss this question for the heat equation and the Hermite-heat equation with Dirichlet boundary control in one dimension. In these two cases, we will see the reachable spaces are spaces of analytic functions which can be described solving complex analysis problems. Finally, I will formulate some open questions and give some perspectives on these questions.

Lassi Paunonen, Tampere U., Finland, *Non-uniform stability of damped contraction semigroups.*

In this presentation we study the stability properties of strongly continuous semigroups generated by operators of the form $A - BB^*$, where A generates a unitary group or a contraction semigroup, and B is a possibly unbounded operator. Such semigroups are encountered in the study of hyperbolic partial differential equations with damping on the boundary or inside the spatial domain. In the case of multidimensional wave equations with viscous damping, the associated semigroup is often not uniformly exponentially stable, but is instead only "polynomially" or "non-uniformly stable". Motivated by such situations, we present general sufficient conditions for polynomial and non-uniform stability of the semigroup generated by $A - BB^*$ in terms of generalised observability-type conditions of the pair (B^*, A). The proofs in particular involve derivation of resolvent estimates for the operator $A - BB^*$ on the imaginary axis. In addition, we apply the results in studying the stability of hyperbolic PDEs with partial or weak dampings.

The research is joint work with R. Chill, D. Seifert, R. Stahn and Y. Tomilov (https://arxiv.org/abs/1911.04804).

Karel Pravda-Starov, U. Rennes 1, France, *Uncertainty principles and null-controllability of evolution equations enjoying Gelfand-Shilov smoothing effects.*

We discuss uncertainty principles for finite combinations of Hermite functions and establish some spectral inequalities for control subsets that are thick with respect to some unbounded densities growing almost linearly at infinity. These spectral inequalities allow to derive the null-controllability in any positive time for evolution equations enjoying specific regularizing effects in Gelfand-Shilov spaces.

This is a joint work with Jérémy Martin (Université de Rennes 1).

Alexander Pushnitski, King's College, London, UK, *Inverse spectral theory for non-compact Hankel operators.*

A version of inverse spectral theory for COMPACT Hankel operators was developed by Patrick Gérard and Sandrine Grellier during 2010-2017. A beautiful feature of this theory is that the spectral map (the map from the set of compact Hankel operators to the set of spectral data) is bijective. Recently, Patrick Gérard, Sergei Treil and myself were able to make further progress in this direction by extending most aspects of this theory to (a subclass of) bounded NON-COMPACT Hankel operators. It turns out that in this case the natural extension of the spectral map is injective, but not surjective. The loss of surjectivity is quite subtle and is related to the appearance of absolutely continuous spectrum in this problem. I will attempt to describe the main features of this work in the talk.

Lionel Rosier, U. du Littorale Côte d'Opale, France, *Exact controllability results of anisotropic 1D equations in spaces of analytic functions.*

Carleman estimates are a very efficient tool to establish the null controllability of parabolic equations, but they are not helpful to decide whether a given state is indeed reachable for such equations. Recently, the determination of the reachable space for the boundary control of the heat equation has attracted the attention of many researchers. In this talk, we shall review some recent results in the linear case (heat equation) and in the semilinear case (semilinear heat equation, anisotropic 1D PDE).

Pascal Thomas, U. Toulouse, France, *Invertibility threshold for Nevanlinna quotient algebras.*

Let $\mathcal{N}$ be the Nevanlinna class and let B be a Blaschke product. It is shown that the natural invertibility criterion in the quotient algebra $\mathcal{N}/B\mathcal{N}$, that is, $|f| \geq e^{-H}$ on the set $B^{-1}\{0\}$ for some positive harmonic function H, holds if and only if the function $-\log|B|$ has a harmonic majorant on the set $\{z \in \mathbb{D} : \rho(z,\Lambda) \geq e^{-H(z)}\}$; at least for large enough functions H. We also study the corresponding class of positive harmonic functions H in the unit disc such that the latter condition holds.

We also discuss the analogous invertibility problem in quotients of the Smirnov class.

This is a joint work with Artur Nicolau.

Yuri Tomilov, Institute of Mathematics, PAN, Poland, *What can spectral theory do for (nonlinear) instability?*

We discuss the norm growth of solutions to difference equations of the form

$$x_{n+1} = Ax_n + K_n(x_n), \qquad x_0 = x, \quad n \in \mathbb{N},$$

where A is a bounded linear operator on a Banach space X, and $K_n : X \to X$ is a compact, in general nonlinear, map for each n. We show that essentially K_n can be treated as linear maps, and the growth of $(\|x_n\|)_{n=0}^{\infty}$ is determined by the spectral properties of A. Moreover, we note that similar results hold in the continuous time setting for abstract differential equations with unbounded operator coefficients. However, the situation here is more involved, and it will only be mentioned briefly.

This is joint work with V. Müller and R. Schnaubelt.

Brett Wick, Washington University, St. Louis, US, *Singular integral operators on the Fock space.*

In this talk we will discuss the recent solution of a question raised by K. Zhu about characterizing a class of singular integral operators on the Fock space. We show that for an entire function φ belonging to the Fock space $\mathscr{F}^2(\mathbb{C}^n)$ on the complex Euclidean space $\mathbb{C}^n$, the integral operator

$$S_\varphi F(z) = \int_{\mathbb{C}^n} F(w)e^{z\cdot\bar{w}}\varphi(z-\bar{w})\,d\lambda(w), \quad z \in \mathbb{C}^n,$$

is bounded on $\mathscr{F}^2(\mathbb{C}^n)$ if and only if there exists a function $m \in L^\infty(\mathbb{R}^n)$ such that

$$\varphi(z) = \int_{\mathbb{R}^n} m(x)e^{-2\left(x-\frac{i}{2}z\right)^2}dx, \quad z \in \mathbb{C}^n.$$

Here $d\lambda(w) = \pi^{-n}e^{-|w|^2}dw$ is the Gaussian measure on $\mathbb{C}^n$.

With this characterization we are able to obtain some fundamental results of the operator S_φ, including the normality, the C^* algebraic properties, the spectrum and its compactness. Moreover, we obtain the reducing subspaces of S_φ.

In particular, in the case $n = 1$, this gives a complete solution to the question proposed by K. Zhu for the Fock space $\mathscr{F}^2(\mathbb{C})$ on the complex plane $\mathbb{C}$ (Integr. Equ. Oper. Theory **81** (2015), 451–454).

This talk is based on joint work with Guangfu Cao, Ji Li, Minxing Shen, and Lixin Yan.